Rate of Convergence of Quasi-periodic Projections

Eva Kopecká

Department of Mathematics, Universität Innsbruck, Innsbruck, Austria
e-mail: eva.kopecka@uibk.ac.at

Abstract

Let $L_1, \dots, L_K$ ($K \geq 2$) be closed linear subspaces in a Hilbert space H, $L_1 \cap \cdots \cap L_K = \{0\}$. Let P_i denote the orthogonal projection onto L_i. The alternating projections $x_n = (P_K \circ \cdots \circ P_1)^n(x_0)$ converge in norm to 0 for any starting point $x_0 \in H$ (von Neumann, 1949, for $K = 2$; Halperin, 1962, for $K \geq 2$). Deutsch and Hundal (2010) asked the following question: do alternating projections converge polynomially fast for starting points from $L_1^\perp + \cdots + L_K^\perp$? We answered the question of Deutsch and Hundal positively in 2020. More precisely, for any $x_0 \in L_1^\perp + \cdots + L_K^\perp$, we have

$$|x_n| \leq \frac{C(x_0)}{\sqrt{n}}, \quad n = 1, 2, \dots,$$

where $C(x_0)$ is a constant depending only on x_0 (Borodin and Kopecká, 2020, for $K = 2$; Reich and Zalas, 2023, for $K \geq 2$). In this estimate, $\sqrt{n}$ cannot be replaced by $n^{1/2+\varepsilon}$ for any $\varepsilon > 0$ even for $K = 2$.

We present a similar result for quasi-periodic consecutive projections.

Theorem. Let $L_1, \dots, L_K$ be closed subspaces of H with $L_1 \cap \cdots \cap L_K = \{0\}$, and let $x_0 \in L_1^\perp + \cdots + L_K^\perp$. Let $\{x_{n+1} = P_{\alpha(n)} x_n\}_{n=0}^\infty$ be a sequence of quasi-periodic consecutive projections onto the family $\{L_1, \dots, L_K\}$ with a quasi-period $M \in \mathbb{N}$. This means, if $I \subset \mathbb{N}$ is an interval of length M then $\{\alpha(n) : n \in I\} = \{1, \dots, K\}$. Then there exists $C(x_0, M) > 0$ such that

$$|x_n| \leq C(x_0, M) \cdot n^{-1/(4M+2)}, \quad n = 1, 2, \dots.$$

Keywords: Consecutive projections, Hilbert space, rate of convergence.

References:

[1] P. Borodin and E. Kopecká, Consecutive Projections and Greedy approximation in Hilbert space. J. Math. Sci. 299 (2026), no. 3, 314–343.