Gaussian and Eisenstein Arithmetic Groups in the Algebra of Complex Quaternions

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Abstract

We introduce two arithmetic groups G and G' arising from the complex quaternion algebra $\mathbb{H}_{\mathbb{C}} = \mathcal{M}_{\mathbb{H}} \oplus i\,\mathcal{M}_{\mathbb{H}}$ and associated with the Gaussian and Eisenstein integer rings. The Gaussian group G is shown to coincide with $GL_2(\mathbb{Z}_i)$, providing an explicit quaternionic description of the latter. In contrast, the Eisenstein group G' defines a distinct arithmetic structure, being neither equal nor isomorphic to $GL_2(\mathbb{Z}_\omega)$. We determine its torsion subgroup of order 24, construct infinitely many elements of infinite order arising from Pell-type equations, and prove that, unlike the Gaussian case, it does not admit a global semidirect product decomposition. We further study the determinant morphism $\xi \mapsto \log|\det(\xi)|$ on $\mathbb{H}_{\mathbb{C}}^{\times}$, which provides a unified interpretation of both groups as arithmetic sections of the determinant kernel and reveals a rigidity phenomenon induced by the quaternionic determinant. Finally, we introduce the determinantal quaternionic theta series

$$\Theta_\kappa(\xi,\eta;s) = \sum_{t\in\mathbb{Z}_\kappa} \exp\left(-s\,|\det(\xi+t\eta)|^2\right), \quad \mathbb{Z}_\kappa \in \{\mathbb{Z}_i, \mathbb{Z}_\omega\}, \quad \Re(s) > 0,$$

associated with quaternionic parameters ξ, η belonging to the Gaussian/Eisenstein quaternionic lattice $\mathcal{M}_{\mathbb{H}}(\mathbb{Z}_\kappa) \oplus i\,\mathcal{M}_{\mathbb{H}}(\mathbb{Z}_\kappa)$. We establish its convergence and invariance properties and prove that it defines a holomorphic function on the half-plane $\Re(s) > 0$.

Keywords: Complex quaternions, arithmetic quaternionic group, Gaussian and Eisenstein integer, torsion subgroup, semidirect product, quaternionic lattice, theta series.

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