# Consistent Parameter Estimation for Stochastic Differential Equations via Theta-Milstein GMM: Theory, Stability Thresholds, and Empirical Validation

Majda El Allali[(1)], Toufik Chaayra[(1)], Mohammed Khellouf[(1)] and Zakaria El Allali[(1)]

[(1)] *Team of MSC, Department of Mathematics, FPN, Mohammed First University in Oujda, Morocco*
*e-mail: majda.elallali@ump.ac.ma*
*e-mail: t.chaayra@ump.ac.ma*
*e-mail: mohammed.khellouf@ump.ac.ma*
*e-mail: z.elallali@ump.ac.ma*

**Abstract**

We establish the $L^2$-consistency of generalized method of moments (GMM) parameter estimators for stochastic differential equations discretized via $\theta$-Milstein schemes. Our key theoretical contribution identifies the unconditional mean-square $A$-stability threshold: $\theta \geq 3/2$, strictly above the classical $\theta \geq 1/2$ for Euler-Maruyama methods. This elevation stems from variance amplification by the second-order Itô correction. We derive explicit GMM estimators exploiting exact conditional moments and prove $L^2$-consistency via moment decomposition. Geometric Brownian motion illustrates lower-order scheme failures: under high volatility, Euler-based maximum likelihood estimation (Euler-MLE) generates negative trajectories in 70.7% of paths at coarse steps. Monte Carlo experiments across three volatility regimes confirm convergence rates and demonstrate substantially lower root-mean-square error (RMSE) than Euler-MLE and standard Milstein schemes. Empirical application to 495 daily S&P 500 prices yields 53% mean-squared error (MSE) reduction versus Euler-MLE, with coefficients of determination exceeding 98.9%.

**Keywords:** $L^2$-Consistency, $\theta$-Milstein scheme, $A$-stability, generalized method of moments, parameter estimation, stochastic differential equations, S&P 500 index

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