



Existence Results for a Neumann Problem with Hardy-Littlewood-Sobolev Critical Exponent and Hardy Potential

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Abstract

We study a class of nonlinear elliptic equations involving a Hardy potential and a critical Choquard-type nonlocal term. The presence of both the inverse-square potential and the critical nonlocal nonlinearity introduces significant analytical difficulties.

Using variational methods, we prove the existence of a nontrivial weak solution by applying the Mountain Pass Theorem. A key step in the analysis is the verification of the Palais–Smale condition below a suitable critical level.

To establish the subcritical nature of the energy level, we construct appropriate test functions and perform asymptotic expansions near the boundary. This allows us to show that the Mountain Pass level lies strictly below the critical threshold.

As a consequence, we obtain the existence of a nontrivial solution to the problem.

Keywords: Elliptic problems, Choquard equations, critical Hardy–Littlewood-Sobolev exponent, subcritical perturbation.

References:

- [1] H. Brézis and L. Nirenberg, Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents. *Comm. Pure Appl. Math.* 36 (1983), no. 4, 437–477.
- [2] E. H. Lieb, Sharp constants in the Hardy-Littlewood-Sobolev and related inequalities. *Ann. of Math.* (2) 118 (1983), no. 2, 349–374.
- [3] V. Moroz and J. Van Schaftingen, A guide to the Choquard equation. *J. Fixed Point Theory Appl.* 19 (2017), no. 1, 773–813.
- [4] J. Giacomoni, S. Rawat and K. Sreenadh, Critical exponent Neumann problem with Hardy-Littlewood-Sobolev nonlinearity. *arXiv:2304.05447* (2023).
- [5] G. Gu and A. Jevnikar, Nonlocal problems with Hardy-Littlewood-Sobolev critical exponent and Hardy potential. *arXiv:2509.15697* (2025).