Fractional-Order Differential Approximations for the Distribution of Finite Sums of $\alpha - \mu$ Random Variables with Application

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Abstract

A novel and mathematically rigorous approximation for the probability density function of the sum of independent and identically distributed $\alpha-\mu$ random variables (RVs) is proposed based on a Fractional-Order Differential Approximation (FODA). Unlike conventional approaches that rely on integer-order moment matching or closed-form Mellin–Barnes integrals, the proposed framework exploits the connection between the fractional derivatives of the characteristic function and the fractional moments of the underlying distribution. By constructing a fractional-order expansion of the characteristic function of each $\alpha-\mu$ RV and combining it analytically to approximate that of the sum, an explicit closed-form expression for the approximate PDF is derived in terms of the univariate Fox's H-function. This new analytical form provides higher flexibility than existing Meijer's G-function or Gamma-based models, as it captures both the central and tail behaviors through tunable fractional parameters. Closed-form expressions for the corresponding cumulative distribution function and moment-generating function are also obtained within the same framework. The method is further extended to the case of independent non-identically distributed $\alpha-\mu$ RVs, maintaining analytical tractability. Monte Carlo simulations confirm the accuracy and robustness of the proposed FODA model under various fading and shadowing conditions, demonstrating excellent agreement with empirical distributions and outperforming conventional integer-order approximations.

Keywords: Fractional-order derivatives, $\alpha - \mu$ distribution, characteristic function, Fox's H-function, probability density function approximation, wireless communication systems.

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