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# An Approximate Formula for the Expected Value of Residual Waiting Time Process under Heavy-Tailed Distributions

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**Abstract**

Let $W(x) = S_{\nu(x)} - x$ be the residual waiting time of a renewal process, $\nu(x) = \inf\{n \geq 1 : S_n > x\}$ and let $R(x) = \mathbb{E}[W(x)]$. For non-negative i.i.d. interarrival times with distribution function $F$, survival function $\overline{F}$, and finite mean $\mu$, the identity $R(x) = \mu U(x) - x$ transfers the bounds for renewal-function $U(x)$ directly to $R(x)$. Chadjiconstantinidis [1] obtained exponential-type bounds involving the root $\kappa(x) > 0$ defined by $\int_0^x e^{\kappa(x)y}\, dF(y) = 1$, which yield

$$\mu \frac{1 - e^{-x\kappa(x)}}{\overline{F}(x)} - x \leq R(x) \leq \mu \frac{1 - e^{-2x\kappa(x)}}{\overline{F}(x)} - x.$$

To facilitate the numerical computation of $\kappa(x)$, we construct an analytical initial bracket using the truncated moments $m_j(x) = \int_0^x y^j\, dF(y)$, $j = 1, 2$:

$$\frac{1}{x} \log\left(1 + \frac{x\overline{F}(x)}{m_1(x)}\right) \leq \kappa(x) \leq \frac{-m_1(x) + \sqrt{m_1^2(x) + 2m_2(x)\overline{F}(x)}}{m_2(x)}.$$

If $[a_n(x), b_n(x)]$ is the interval obtained after $n$ bisection steps, its midpoint is defined by $\widehat{\kappa}_n(x) = (a_n(x) + b_n(x))/2$. Substituting $\widehat{\kappa}_n(x)$ into the mean of the two bounds gives the following approximation for expected value of residual waiting time process $R(x)$:

$$R_{\mathrm{bapp},n}(x) = \frac{\mu}{2\overline{F}(x)} \left[2 - e^{-x\widehat{\kappa}_n(x)} - e^{-2x\widehat{\kappa}_n(x)}\right] - x.$$

The method is illustrated for the heavy-tailed Pareto/Lomax distribution with shape $\alpha = 2$ and scale $\lambda = 1$, and compared with the numerical values reported in [1]. Numerical comparisons based on the values reported in [1] demonstrate the applicability of the method in a heavy-tailed setting.

**Keywords:** residual waiting time, renewal process, exponential-type bounds, bisection method, heavy-tailed distributions.

## References:

[1] S. Chadjiconstantinidis, Exponential and Pareto-type bounds for the renewal function and the excess lifetime of a renewal process, *Statistics* 58(6) (2024), 1443–1462.