



New Enclosures for Bessel Functions

Ulrich Reif

*Technical University of Darmstadt, Department of Mathematics, Darmstadt, Germany
e-mail:reif@mathematik.tu-darmstadt.de*

Abstract

We present a new paradigm for deriving enclosures for Bessel functions J_ν and Y_ν of the first and second kind for real arguments and orders. It is based on a hitherto unknown monotonicity property of a differential operator related to the modulus function $M_\nu := \sqrt{J_\nu^2 + Y_\nu^2}$. When applied to the asymptotic expansions of the modulus M_ν and the phase $\Theta_\nu := \arctan Y_\nu/J_\nu$, we obtain significantly tighter enclosures than the known ones. Moreover, former results for the asymptotic expansion of M_ν apply only if the number of summands exceeds the order ν and results for the asymptotic expansion of Θ_ν apply only if $\nu < 3/2$. By contrast, our findings apply to any number of summands and arbitrary orders $\nu \geq 0$, thus extending the covered range substantially.

Further, we introduce the most simple, yet powerful one-term approximation

$$M_\nu(x) \approx \sqrt{\frac{2}{\pi \sqrt{x^2 - \nu^2 + 1/4}}}$$

of the modulus. We discuss its convergence properties, specify enclosures for itself and the resulting approximation of the phase Θ_ν , and conclude with some remarks concerning convergent bounds on the location of the zeros of J_ν and Y_ν .

Keywords: Bessel function, asymptotic expansion, enclosure.